Variable Decomposition in Total Variant Regularizer for denoising/deblurring Image

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***Abstract*-The aim of image restoration is to obtain a higher quality desired image from a degraded image. In this strategy, an image inpainting methods fill the degraded or lost area of the image by appropriate information. This is performed in such a way so that the resulted image is not distinguishable for a casual person who is not familiar with the original image. In this paper, the various images are degraded with different ways: 1) the blurring and adding noise in the original image, and 2) losing a percentage of the pixels of the original image. Then, the proposed method and other methods are performed to restore the desired image. It is required that the image restoration method use optimization methods. In this paper, a linear restoration method is used based on the total variation regularizer. The variable of optimization problem is decomposed, and the new optimization problem is solved by using Lagrangian augmented method. The experimental results show that the proposed method is faster, and the restored images have higher quality than other methods.**

Keywords- Image restoration; Image inpainting; Deblurring; Total variation regularizer; Lagrangian augmented.

I. INTRODUCTION

Image restoration is one of the most important image processing techniques. Image restoration is used in various applications and areas such as medical, astronomical imaging, image and video coding, remote sensing, military, seismography, aerology, film restoration, and so on [1]. In space exploration, image restoration systems have been used by researchers since 1960 [2]. Providing the desired image from ruined image is the aim of the image restoration systems. The image restoration system contains de-blurring, de-noising, and preserving fine details [3]. The information and details of image are lost during image capture. Not only, restoration removes the noise of images but also is widely used in the blind deconvolution, the image inpainting, and various image processing methods [4, 5, 6]. The image inpainting is a process of reconstructing corrupted or lost parts of the image that is not distinguishable for a casual person who is not familiar with the original image. The image inpainting plays important role in the various image processing applications such as removal of scratches in old photographs and video, filling in missing blocks in unreliably transmitted images, and removal of overlaid text or graphics [5]. Image demolition causes from non-adjusted camera, object and camera motion, reflection from uncontrollable sources and non-ideal photographic and communication systems [5]. The most common problem in photography is the image blurring and noise. The blurring occurs because of a localized averaging of pixels, and significant in light limited situations and resulting in a ruined photograph. Image deblurring is the process of recovering a sharp image from the corrupted image. The blurring contains environmental blurs and motion blurs. The reason of the environmental blurs is a light passing through the media environment with different refractive index. The motion blur causes from relative motion between camera and scene [7]. In this paper, it is assumed that the blur motions are distinguishable and estimable, and noise is Gaussian distribution with zero mean.

The image restoration system includes three important parts: a) modeling the degraded image, b) formulating the image restoration problem c) designing a new efficient and accurate method to solve the image restoration problem.

In modeling part, the blurring and noise information are used to create a model of the degraded image. In the many recent researches, the linear model is used for modeling the degraded image. Common degradations include noise, blurring, color imperfections and geometrical distortions. The image restoration problems can be modeled by using the following expressed linear degradation model [1] :

$$y = Bx + n \, , \tag{1}$$

where B is a point spread function (PSF), x is the original image, n is noise vector, and y represents the degraded image. If the PSF is specific (the PSF is the same for all image pixels), the Eq. (1) is identified deconvolution problems, and if the PSF is not specific (the PSF changes throughout the image), the Eq. (1) is identified blind deconvolution problems.

In formulating part, the information of degraded and original images are used to formulate the objective function and optimized to remove the noise and blurring from degraded image. This function is solved by using the inverse function or optimization problem. The image restoration problem that is solved by convex optimization uses unconstrained optimization as follow:

$$\min_x \frac{1}{2}\|y - Bx\|_2^2 + \tau\phi(x) \, , \tag{2}$$

where B is a linear operator, $\phi(x)$ is a regularizer and $\| \; \|_2$ is 2-norm that obtained by[8]:

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$$\|X\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}. \quad (3)$$

The optimization problem contains two parts: data fidelity and smooth regularizer. In practical researches, the regularizer cannot completely model the characteristic of original image. Therefore, it is required to compromise between regularizer and data fidelity.

II. RELATED WORKS

The image restoration methods are classified to three categories: 1) the methods based on filtering 2) the methods based on regularizer, and 3) the methods based on Bayesian restoration.

The image processing systems use low pass filter to model the blurring of the image. The filter-based methods contain the inverse filtering, the pseudo-inverse filtering and the wiener filtering methods [2, 9,10].

The methods based on regularizer are the one of the three categories of the image restoration methods. The regularizer method repeatedly combines additional information and a regularizer to solve restoration problem. These methods, such as Tikhonove-Miller regularizer, are modified to make the restoration problem well posed. The traditional regularizers, such as L2-norm, adversely affect the sharp edges restoration, because the images are piecewise smooth. Therefore, the advanced regularizers model the characteristics of original image by using nonlinear penalty functions [11].

The Bayesian methods combine the additional information of a new image and the prior image and iteratively can be solved which are the advantages of the Bayesian methods. High computational cost and not providing a specific optimization framework are disadvantages of the Bayesian methods [12].

In this paper, the regularizer method is used to restore the image. There are many methods to solve the linear inverse optimization as minimizing the objective function given by Eq. (2). This optimization equation should find the best compromise between candidate estimated, x, and the obtained data of $\|y - Bx\|_2^2$. The undesired degree of equation is distinguished by $\phi(x)$ parameter, and the relation between two parts of Eq. (5) is explained by regulating parameter (τ). The unsmooth and non-quadratic regularizers, such as the Total Variant (TV) and l_p-norm, are used in various image processing applications [9]. If B = I, means that I is a identity vector, the denoising problem is confronted. If $\emptyset$ is suitable and convex, the optimization problem is strictly convex and has unit minimizer. Therefore, the denoising function is explained by

$$\psi_\tau(u) = \arg\min_x \frac{1}{2}\|z - u\|_2^2 + \tau\phi(u). \quad (4)$$

For example, $\psi_\lambda(u) = \text{soft}(u, \tau)$ when $\phi(u) = \|u\|_1$ The iterative shrinkage thresholding (IST) methods have been proposed to efficiently and simply solve the sparsity-based restoration problems [13, 14]. In the IST method, x in k+1 step is obtained by

$$x^{k+1} = \psi_{\tau/\alpha}(x^k - \frac{1}{\alpha_k}\left(B^T\left(Bx^k - y\right)\right) \quad (5)$$

These methods are suitable and efficient when multiplication of B and B^T is dissolvable. These methods are converged to minimum when $\|B\|_2^2 / 2 < a_k < +\infty$.

The next method is two-step iterative shrinkage thresholding (TwoIST) in which each current iterate incorporates to the previous two iterates. The TwoIST method is faster than the IST method. This method compensates the weakness of the IST method by applying the two-step nonlinear iterating of the IST method. Therefore, the convergence rate of the TwoIST method is more than the convergence rate of the IST method [15].

The next two step variant of the IST methods is Fast IST Algorithm (FISTA). FISTA is so faster than TwoIST and IST methods. The non-smooth variation of Nesterovs optimal gradient-based algorithm is used in the FISTA [16].

Sparse Reconstruction by Separable Approximation Algorithm (SpaRSA) is another fast variant of IST algorithm. This method uses a different α_k in each iteration that is updated by: $\alpha_k \leftarrow \eta\alpha_k$ [17].

Although in [18, 19] it was reported to use augmented lagrangian for total variation, in this paper it is proposed a new method based on variable decomposition which results in the computation time is lower than the one by the aforementioned methods where the quality of restored image is higher.

III. PROPOSED METHOD

Image restoration problem is more accurately expressed using non-linear regularizer [11, 15]. Note that non-linear can be also considered as TV. Therefore, in this paper, it is tried to use these regularizers.

In Eq. (1) , x and y indicate the original and degraded image, respectively, and B is a linear operator which induces a blur vector in one case or losing a number of image pixels in another case. In order to construct the degraded image, it needs to add the linear operator of B and noise to the original image. Depending on the linear operator, B, includes the blur or the lost pixels, the image restoration problem will differ. If B contains the blur, the image restoration problem changes to the deblurring and if B indicates the lost pixels, it converts to the inpainting.

A. Construction of the degraded images for deblurring and inpainting

Deblurring tries to remove the blur and noise existent in degraded image. Assume that the aim is to degrade an image with size of N×N by m×m uniform blur where m<N. First, a vector with length of N is constructed so that the first m elements have the value of 1/m and the rest elements are zero. Next, the obtained vector is shifted to left with size of (m-1)/2 and then it is multiplied by its transpose. Therefore, N×N Matrix is constructed which is called the blur matrix. In order to obtain the blurred image, it requires to multiply the Fourier transform of the blurred matrix by the Fourier transform of the original image and then to use inverse Fourier transform. The obtained matrix is Bx

indicated in Eq. (1) next, a desired noise is added to Bx, and therefore the blurred/noisy image is constructed. In order to construct degraded image for inpainting case, it require losing a number of pixel. Losing pixels are randomly performed. To obtain the degraded image in Eq. (1), the random matrix is constructed such that a percentage of the original image pixels are lost. Note that the size of random matrix is as same as the original image.

A. B. Solving the problem of Lagrangian augmented optimization

In order to obtain the estimated image, x, Eq. (2) is used the proposed method based on variable decomposition for optimization problem. The target function in Eq. (2) is sum of two functions. The main idea in the proposed method is to decompose the variable of x into variable pairs of x and v such that each of them is an argument of one part in the target function. Then, the target function is minimized under one constraint which results in being equally the new problem with the problem in Eq. (2) that is given by

$$\min_{x,v\in\mathbb{R}^n} \frac{1}{2}\|Bx-y\|_2^2+\tau\phi(v) \quad \text{subject to } v=x \tag{6}$$

The new determined optimization problem is solved by lagrangian augmented. Consider the formulation of undetermined optimization for regulated image restoration which is given by

$$\begin{aligned} f_1(x) &= \frac{1}{2}\|Bx-y\|_2^2 \\ f_2(x) &= \tau\phi(x) \\ y&=Iu \quad, G=I \end{aligned} \tag{7}$$

The formulation of the determined optimization using variable decomposition is given by

$$\text{objective fun} = \min_{x,v\in\mathbb{R}^n} f_1(x)+f_2(v) \quad \text{subject to } v=x \tag{8}$$

In order to solve the determined optimization problem, it is better to use Lagrangian augmented method which is given by

$$\begin{aligned} (x_{k+1},v_{k+1}) &\in \arg_{x,v} \frac{1}{2}\|Bx-y\|_2^2+\tau\phi(v)+\frac{\mu}{2}\|x-v-d_k\|_2^2, \\ d_{k+1} &= d_k-(Gx_{k+1}-v_{k+1}) \end{aligned} \tag{9}$$

where μ is a positive value which is given by user. It has been shown the performance of least-squares penalty is better than Lagrangian method. In addition the Lagrangian augmented is converged under more principle conditions. Therefore, a new method is proposed to solve the optimization problem as follows.

First, k is set to zero, $\mu>0$, d_0 and v_0 are set to initial value, then, it requires to solve the optimization problem given by:

$$x_{k+1} = \arg\min_x \|Bx-y\|_2^2+\mu/2\,\|x-v_k-d_k\|_2^2, \tag{10}$$

Here, x_{k+1} is a strictly convex function which has to be minimized. This corresponds to a linear system which is given by

$$x_{k+1} = \left(B^H B+\mu I\right)^{-1}\left(B^H y+\mu\left(v_k+d_k\right)\right). \tag{11}$$

By obtaining x_{k+1} from previous step, it is inserted in Eq. (12) and therefore it requires solving the optimization problem as:

$$v_{k+1} = \arg\min_v \tau\phi(v)+\frac{\mu}{2}\|x_{k+1}-v-d_k\|_2^2. \tag{12}$$

This can be solved by Eq. (4). Then, d_{k+1} is obtained by x_{k+1} and v_{k+1} previous step,

$$d_{k+1} = d_k-\left(x_{k+1}-v_{k+1}\right). \tag{13}$$

Next, one is added to k (k=k+1) and stop a criterion is measured which is given by

$$\frac{|\text{objective fun}(k+1)-\text{objective fun}(k)|}{|\text{objective fun}(k)|} < \text{tolerance}. \tag{14}$$

Where

$$\text{objective fun}(k)=\frac{1}{2}\|Bx_k-y\|_2^2+\tau\phi(v_k) \tag{15}$$

This criterion is equal to the changes of the target function. If the stop criterion is satisfied, the procedure is stopped; otherwise the previous steps are repeated until the stop criterion is satisfied. The variation is defined by

$$\phi(u) = \sum_1^l \sqrt{\nabla_x(u_i)^2+\nabla_y(u_i)^2}, \tag{16}$$

where $\nabla_x(u_i)$ and $\nabla_y(u_i)$ are first order vertical and horizontal difference in i^{th} pixel, respectively.

B.1- Lagrangian augmented

Consider the optimization problem with constraint as follows:

$$\min_{z\in\mathbb{R}^n} E(z) \quad s.t\ Hz-b=0 \tag{17}$$

In this case, Lagrangian function is given by:

$$\mathcal{L}_A(z,\lambda,\mu) = E(z)+\lambda^T(b-Hz)+\mu/2\,\|Hz-b\|_2^2, \tag{18}$$

where λ is a value of Lagrangian coefficient and $\mu>0$ is called penalty parameter. In this method, $\mathcal{L}_A(z,\lambda,\mu)$ is minimized with respect to z and maintaining λ as a constant value, then, λ is updated and the minimization of $\mathcal{L}_A(z,\lambda,\mu)$ is repeated. This procedure continues until the convergence criterion is satisfied [20].

B.2- Variable decomposition using Lagrangian augmented

Consider the variable decomposition problem given by

$$z=\begin{bmatrix}u\\v\end{bmatrix}, H=[G\ \ -I],\ b=0 \qquad (19)$$
$$\min\ R(u)+\tau\phi(v)\quad s.t\quad Hu=v$$

In this case, Lagrangian augmented is given by

$$\mathcal{L}_A(u,v,\lambda)=\mathcal{R}(u)+\tau\phi(v)+\lambda^T(Gu-v)+\mu/2\|V-Gu\|_2^2, \qquad (20)$$

where λ is Lagrangian coefficients vector and μ is a constant which is selected by a user. Gauss–Seidel method is used for minimization. Therefore, the minimization problem is formulated as,

$$u_{k+1}=\arg\min_u\ \mathcal{L}_A(u_k,v_k,\lambda_k), \qquad (21)$$

$$v_{k+1}=\arg\min_v\ \mathcal{L}_A(u_{k+1},v_k,\lambda_k), \qquad (22)$$

$$\lambda_{k+1}=\lambda_k+\mu(Gu_{k+1}-v_{k+1}) \qquad (23)$$

$$\text{if } d=\frac{\lambda}{\mu},\ \ \lambda^T r+\frac{\mu}{2}\|r\|_2^2=\frac{\mu}{2}\left\|r+\frac{1}{\mu}\lambda\right\|_2^2-\frac{1}{2\mu}\|\lambda\|_2^2$$

$$\mathcal{L}_A(u,v,d)=\mathcal{R}(u)+\tau\Phi(v)+\mu/2\|v-Gu+d\|_2^2$$

B.3-Calculation of x_{k+1}

In Eq. (11), the initial values of x_0, d_0 and v_0 are zero. Thus, in Eq. (16), $\phi(v_0)$ which is related to the variations regularizer, can be solved and $\|Bx_0-y\|_2^2$ can be calculated using Eq. (3). The initial target function is calculated by using $\phi(v_0)$ and $\|Bx_0-y\|_2^2$. In following the calculation of x_{k+1} is distinctly explained for deblurring/inpainting and inpainting problems.

- *Calculation of* x_{k+1} *for deblurring/denoising:* First, the absolute of Fourier transform of blur matrix is obtained and each element is squared and added with μ. The obtained matrix elements are then inversed. Next the Fourier transform of $B^H y+\mu(v_k+d_k)$ is obtained and multiplied by $1./(\text{abs}(\text{fft2}(B)).^2+\text{mu})$. x_{k+1} is the inverse Fourier transform of the resulted matrix.

- *Calculation of* x_{k+1} *for inpainting*: In order to obtain the inverse of $(B^H B+\mu I)$, Sherman-Morrison-Woodbury equation is used which is given by

$$\left(B^H B+\mu I\right)^{-1}=\frac{1}{\mu}\left(I-\frac{1}{\mu+1}B^H B\right), \qquad (24)$$

where $B^H B$ is a number of zeros in main diagonal. These zeros indicate the lost position in the image. Thus, x_{k+1} is obtained by multiplication of $B^H y+\mu(v_k+d_k)$ in Eq. (24).

- Calculation of v_{k+1}: The v_{k+1} can be calculated by Moreau proximal mapping for $x_{k+1}-d_k$ .this means that

$$v_{k+1}=\psi(x_{k+1}-d_k), \qquad (25)$$

where ψ is given by Eq. (4). If this mapping is accurately calculated in closed form then it is guaranteed that the proposed method is converged.

IV. RESULTS

All experiments have been executed using MATLAB software. The value of μ in Eq. (10) has been selected as 10% of regularizer parameter or $\tau/10$. The number of iteration, the processing time (CPU time), ISNR and MSE have been used as evaluation measures for different blurs and different percentage of lost pixels. Various images such as: cameraman, Lena, moon, lifting-body, tire, coins, and peppers are used. For blurring, for instance, a uniform blur with size of 9×9 and white normal Gaussian noise are used, and for inpainting, for instance, 40% of pixel related to the original image has been lost.

B. A- Evaluation measures

In order to compare the performance of different image restoration method, quantitative measures which evaluation the quality of restored image are very important. These measures are improvement in SNR, ISNR, and Mean Square Error, MSE that are obtained by [2, 10].

$$ISNR=10\log_{10}\frac{\sum_K\|x-y_k\|^2}{\sum_K\|x-\hat{x}_k\|^2}, \qquad (26)$$

$$MSE=\frac{1}{M*N}\sum_K\|x-\hat{x}_k\|^2, \qquad (27)$$

where M and N are image dimensions, x is the original image, and y_k and $\hat{x}_k$ are the observed and the estimated image in k^{th} iteration. In this paper, besides on these two measures, processing time has been also considered which indicates the capability of the methods.

C. B- Indexing results

B-1-The obtained results for deblurring

Table 1 shows the evaluation measures for the aforementioned images. The obtained results show that the proposed method improves ISNR, decreases MSE and reduces the processing time, considerably, compared to TwoIST and SpaRSA method. For instance, for lena image, ISNR has been improved by the proposed method 1.03 dB and 1.23dB compared to TwoIST and SpaRSA methods, respectively, and the processing time is nearly 25% of the one by TwoIST and SpaRSA methods.

Table 1. The results of deblurring

Image	Method	Iterations	CPU time(s)	ISNR(dB)	MSE
Cameraman	TwoIST	69	16.2	7 .63	94 .1
	SpaRSA	123	25.7	7 .86	89 .2
	Proposed method	20	3 .67	8 .43	78 .2
Lena	TwoIST	46	51 .1	6 .56	37 .4
	SpaRSA	56	52	6 .36	39 .1
	Proposed method	16	13 .1	7 .59	29 .5
Lifting-body	TwoIST	41	45 .3	8 .64	12 .3
	SpaRSA	53	49 .5	8 .88	11 .6
	Proposed method	24	20 .3	10 .4	8 .14
Coins	TwoIST	61	13 .8	8 .87	54 .4
	SpaRSA	115	23 .1	8 .71	56 .5
	Proposed method	22	4 .06	9 .75	44 .5
Moon	TwoIST	25	13 .1	3 .86	62 .8
	SpaRSA	22	9 .14	3 .73	70 .2
	Proposed method	17	6 .18	3 .91	67 .4
Peppers	TwoIST	51	55 .5	7 .93	31 .8
	SpaRSA	78	73 .4	7 .97	30 .1
	Proposed method	18	14 .8	8 .45	27

In addition, the obtained MSE by the proposed method decreases 10 unit. For liftingbody image, the processing time required by the proposed method is nearly half of the one by the other methods and ISNR has improved around 1.5dB. In addition, the MSE decreases 3.46 And 4.16 compared to SpaRSA and TwoIST methods, respectively.

Fig. 1 shows the target function indicated in Eq. (5) for the aforementioned images. As observed, the target function compared to SpaRSA and TwoIST methods is faster convergence. For instance, for cameraman, the target function is converged to the final value in less than 4 seconds whereas 30 second is required for two methods at least. For lena image, the proposed method is converged in 3 seconds whereas SpaRSA and TwoIST methods need 30s and 50s, respectively, for convergence. The same results can be observed for other images.

B.2- The obtained results for inpainting

In the inpainting problem, the aim is restoration of an image in which a percentage of its pixel has been lost and noise has been added. In this experiment, 40% of image pixels have been lost and additive normal white Gaussian noise has been used. For evaluation, the aforementioned images are used and the proposed method is compared to TwoIST and FISTA methods. The obtained results are shown in Table 2.As observed, the proposed method results in higher ISNR, lower MSE and also shorter processing time processing time compared to TwoIST and FISTA methods. For instance, for Lena image, the processing time is almost 10% of the one provided by two other methods and improves ISNR and MSE al least 0.3 dB and 0.6 dB, respectively.

For lifting-body image, the processing time is around 1000s and 845s less the ones obtained TwoIST and FISTA methods, respectively. Also the ISNR obtained by the proposed method is 0.64dB and 0.97dB higher than the ones resulted by TwoIST and FISTA methods, respectively.

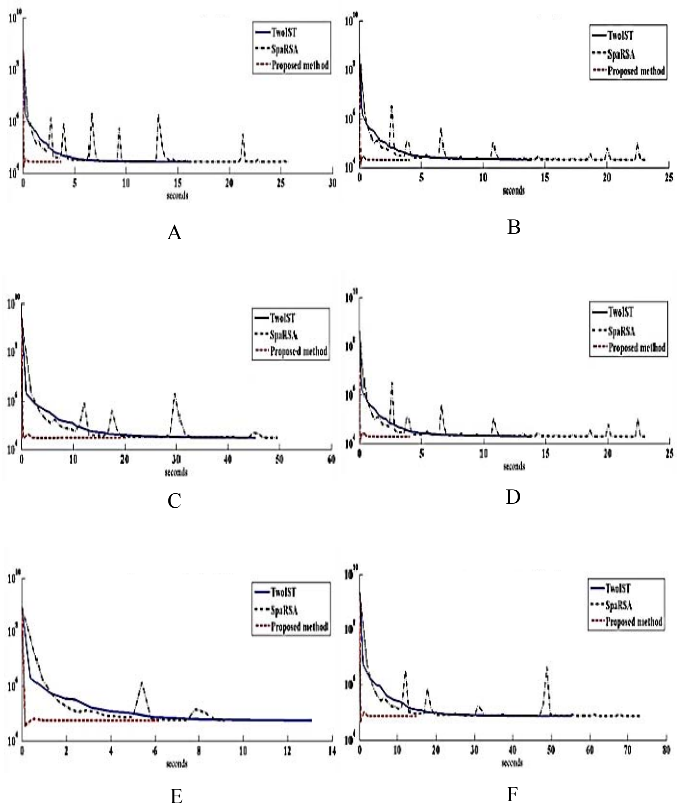

Fig. 1. The Output of deblurring, objective function related to Eq. (5) of TwoIST , SpaRSA and the proposed method for images A) Cameraman , B) Lena, C) Lifting-body, D) Coins, E) Moon , F)Peppers

In this case, the proposed method provided 0.4 and 0.5 reduction in MSE compared to FISTA and TwoIST methods, respectively. The same results for other images can be also concluded.

Table 2. The results of inpainting

Image	Method	Iterations	CPU time(s)	ISNR(dB)	MSE
Cameraman	TwoIST	502	313	18.8	95.6
	SpaRSA	500	194	19	90.7
	Proposed	73	27.5	19.2	85.7
Lena	TwoIST	502	1340	24.9	22.7
	SpaRSA	500	915	25.2	21
	Proposed	55	100	25.5	19.8
Lifting-body	TwoIST	502	1080	7.96	30.2
	SpaRSA	500	925	7.64	30.3
	Proposed	44	79.7	8.61	29.8
Coins	TwoIST	502	214	17.4	112
	SpaRSA	500	184	18.1	95.6
	Proposed	78	25.9	18.3	90.9
Moon	TwoIST	502	1059	21.7	37
	SpaRSA	500	1499	21.9	35.3
	Proposed	127	256	23.1	33.5
Tire	TwoIST	502	105	15.3	85.1
	SpaRSA	500	89	16	71.5
	Proposed	63	11.3	16.7	61.9
Peppers	TwoIST	502	803	24.1	26.7
	SpaRSA	500	559	24.3	25.7
	Proposed	61	72.6	24.8	23.2

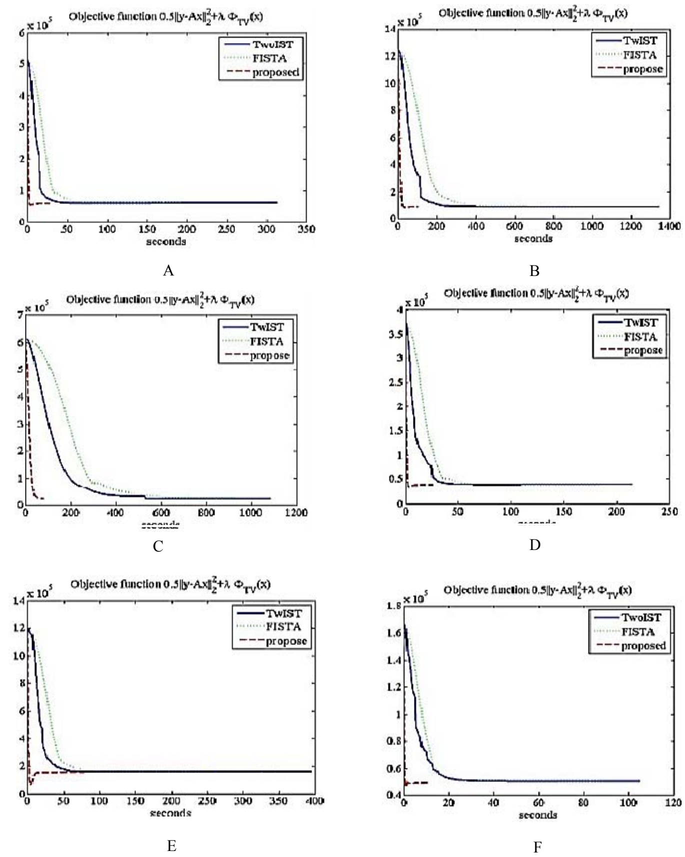

Fig. 2. The Output of inpainting, objective function related to Eq. (5) for TwoIST , FISTA and the proposed method for images A) Cameraman , B) Lena, C) Liftingbody, D) Coins, E) Moon , F)Tire.

Fig. 2 shows the target function resulted by inpainting for the aforementioned images. As observed, the final value in shorter time compared to TwoIST and FISTA methods. For instance, in Fig. 2(a) which is related to cameraman image, the target function converges in 50s, 200s and 300s for the proposed method, TwoIST, and FISTA methods, respectively. The same results can be observed for other images. Therefore, the target function by the proposed method converges faster than other methods.

V. CONCLUSION

In this paper a new image restoration method was proposed. The proposed method used variable decomposition based on total variant regularizer to solve the optimization problem more rapidly. In the optimization problem, since the objective function includes two terms; one term is second order and the other one is non-linear regularizer, the variable decomposition was used. This caused that the argument of each term includes an individual variable. In order to equal the new optimization problem with the initial one, the undetermined optimization problem was converted to the determined one and then the new problem was solved using Lagrangian augmented. Since image piecewise smoothed, traditional regularizers such as L2-norm affect restoration of edge sharpness and smooth the edges. Therefore, the total variant regularizer, due to maintaining the sharpness of the edges and removing the additive noise, was used. The image restoration was applied *Abstract*-The aim of image restoration is to obtain a higher.

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