Time-varying formation control for unmanned aerial vehicles with switching interaction topologies

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1. Introduction

Formation control of unmanned aerial vehicle (UAV) swarm systems has drawn considerable attention from the scientific and engineering communities in recent years. This is partially due to the broad potential applications of formation control for UAV swarm systems in various fields, such as surveillance (Nigam, Bieniawski, Kroo, & Vian, 2012), source seeking (Han & Chen, 2014), drag reduction (Williamson et al., 2007), and telecommunication (Kroo, & Vian, 2012), and telecommunication relay (Sivakumar & Tan, 2010). In fact, formation control is not a new research topic and has been studied a lot in robotics community in the past decades. There are three typical formation control approaches in robotics community; that is, leader–follower based approach (Desai, Ostrowski, & Kumar, 1998; Li & Xiao, 2005), virtual structure based approach (Lewis & Tan, 1997) and behavior based approach (Balch & Arkin, 1998). These approaches have been applied to solve the formation control problems for UAV swarm systems recently.

Based on leader–follower approaches, Wang, Yadav, and Balkrishnan (2007) proposed a hierarchical control strategy for UAV swarm systems to achieve constant formations while avoiding obstacles. Gu et al. (2006) investigated formation control problems for UAV swarm systems using leader–follower approaches and presented formation flight experiment with two fixed-wing UAVs. Yun, Chen, Lum, and Lee (2010) applied leader–follower approaches to unmanned helicopter swarm systems and showed experimental formation results using two single-rotor helicopters. Experimental results on quadrotors were given in Mercado, Castro, and Lozano (2013) using leader–follower based formation control approaches. The effects of switching topologies on the leader–follower based formation control of swarm systems were investigated in Mesbahi and Hadaegh (2001) and Wang and Wu (2012). Linorman and Liu (2008) discussed formation control approaches for UAV swarm systems using virtual structure approaches. Based on virtual structure strategy, Kushleyev, Mellinger, and Kumar (2012) showed a series of indoor formation flight experiments for quadrotor swarm systems. Bayezit and Fidan (2013) proposed a virtual structure based formation controller for UAV swarm systems moving in three-dimensional space and verified the controller via a number of simulations. Dydek, Annsawamy, and Lavretsky (2013) studied formation control problems for UAV swarm systems in the presence of parametric uncertainties and demonstrated the theoretical results using three quadrotors. Formation problems for UAV swarm systems with nonlinear dynamics were addressed in Cruz and Carelli (2008), Kladis, Menon, and Edwards (2011), Kladis (2014), and Smyznakis, Kladis, and Aitken (2015) using a tracking approach. Behavior based formation control problems for UAV swarm systems were studied in Sharma and...
Although leader–follower, virtual structure and behavior based approaches can be used to deal with the formation control problems of UAV swarm systems, Beard, Lawton, and Hadaegh (2001) have pointed out that these approaches have their own weaknesses. For example, leader–follower based formation approaches lack of robustness due to the existence of the explicit leader, and virtual structure based formation approach are not suitable for distributed implementation as it requires lots of communications and computations. In recent years, consensus problems of swarm systems have been studied extensively and many results have been obtained (Jadbabaie, Lin, & Morse, 2003; Manfredi, 2013; Menon & Edwards, 2009; Olfati-Saber & Murray, 2004; Ren & Beard, 2005; Xiao & Wang, 2007; Xi, Shi, & Zhong, 2012). With the development of consensus theory, more and more researchers realized that formation control problems can be solved using consensus based approaches. Lin, Francis, and Maggiore (2005) proposed necessary and sufficient conditions for robot swarm systems to achieve formations. Ren (2007) applied a consensus protocol to deal with the formation control problems of second-order swarm systems and showed that the traditional leader–follower, virtual structure and behavior based formation approaches can be regarded as special cases of consensus based approaches. Moreover, the weakness of the traditional formation control approaches can be overcome.

By combining consensus protocols with attraction/repulsion functions, Jia and Wang (2014) studied the formation control problems for robotic fish swarm systems. For UAV swarm systems to achieve time-invariant formations, Abdessameud and Tayebi (2011) presented a consensus based formation controller with time delays for UAV swarm systems. A consensus approach and an output feedback linearization method were used together to investigate the formation control problems of UAV swarm systems in Seo, Kim, Kim, and Tsourdos (2012). The theoretical results in Abdessameud and Tayebi (2011) and Seo et al. (2012) were validated by numerical simulations. Indoor time-invariant formation flight experiments for quadrotor swarm systems can be found in Turpin, Michael, and Kumar (2012). It should be pointed out that the formation in Ren (2007), Jia and Wang (2014), Abdessameud and Tayebi (2011), and Turpin et al. (2012) are time-invariant. Dong, Yu, Shi, and Zhong (2015) studied time-varying formation control problems for UAV swarm systems and showed experimental formation flight results using five quadrotors. However, in Abdessameud and Tayebi (2011), Seo et al. (2012), Turpin et al. (2012), and Dong et al. (2015), it is assumed that the interaction topologies among the UAVs are fixed. It is well-known that in practical applications, the interactions among UAVs may be switching due to the failure and new creation of the communication link partly caused by the communication range constraints. Therefore, it is significant to consider the effects of switching topologies on the time-varying formation control of UAV swarm systems.

This paper studies time-varying formation control problems for UAV swarm systems with switching interaction topologies: (1) under what conditions the time-varying formation can be achieved, and (2) how to design the formation control protocol. A two-loop formation control configuration is applied, where the inner-loop controller stabilizes the attitude, and the outer-loop controller, which is the main concern of the current paper, drives the UAVs towards the desired positions. A distributed formation control protocol is constructed using the neighboring information of each UAV, where the formation can be time-varying. Then necessary and sufficient conditions for UAV swarm systems to achieve time-varying formations under switching topologies are presented. An approach to determine the gain matrices in the protocol is given by solving an algebraic Riccati equation. Moreover, an explicit expression of the formation reference function is obtained to describe the macroscopic movement of the whole UAV formation. An approach to specify the motion modes of the formation reference is also given. Finally, a formation platform consisting of four quadrotors is introduced. Time-varying formation applications on the quadrotor platform are given in both simulation and experiment to demonstrate the theoretical results.

Compared with the previous results on formation control of swarm systems, the main contributions of the current paper are threefold. Firstly, the formation in this paper is specified by time-varying piecewise continuously differentiable vectors. In Abdessameud and Tayebi (2011), Seo et al. (2012), Turpin et al. (2012), Wang and Wu (2012), and Lin et al. (2005), the formation is specified by time-invariant vectors. Note that time-varying formation will bring the derivative of the formation information into the analysis and design, the results for time-invariant formations in Abdessameud and Tayebi (2011), Seo et al. (2012), Turpin et al. (2012), Wang and Wu (2012), and Lin et al. (2005) cannot be applied to the time-varying case directly. Although the formation in Mesbahi and Hadaegh (2001) is time-varying, it is required that the time-varying formation is twice differentiable, which is more conservative than the formation in this paper. Secondly, the interaction topologies can be switching and the criteria for swarm systems to achieve formation are both necessary and sufficient. However, the interaction topologies in Abdessameud and Tayebi (2011), Seo et al. (2012), Turpin et al. (2012), Dong et al. (2015), Kladis (2014), Kladis et al. (2011), Menon and Edwards (2009), Smyznakis et al. (2015), Li and Xiao (2005), Lin et al. (2005), and Cruz and Carelli (2008) are assumed to be fixed. As shown in Ni and Cheng (2010), cooperative control problems for multi-agent systems with switching topologies are much challenging than the fixed ones. Although the topologies in Mesbahi and Hadaegh (2001) and Wang and Wu (2012) are switching, only sufficient conditions were presented. Thirdly, the formation controller is robust and distributed, and has good scalability as there exist no explicit leaders and it only needs the neighboring information. The roles of all agents in the swarm systems are identical in the current paper. In Wang et al. (2007), Gu et al. (2006), Yun et al. (2010), Mercado et al. (2013), Li and Xiao (2005), Mesbahi and Hadaegh (2001), and Wang and Wu (2012), the formation control problems were discussed under the leader–follower framework, where there exist explicit leaders and the failure of the leader may result in the collapse of the whole formation. Moreover, only one leader and one follower were considered in Mercado et al. (2013), which means that the scalability of the results in Mercado et al. (2013) cannot be guaranteed. The formations in Linorman and Liu (2008), Kushleyev et al. (2012), Bayezit and Fidan (2013), Dydek et al. (2013), Cruz and Carelli (2008), Kladis (2014), Kladis et al. (2011), and Menon and Edwards (2009) are not implemented in a fully distributed way as each agent in the swarm system should track the waypoints calculated centralized, which consume too many communication and computation resources.

The rest of this paper is organized as follows. In Section 2, the problem formulation is shown. In Section 3, time-varying formation control problems are transformed into asymptotic stability problems. Main theoretical results are proposed in Section 4. In Section 5, a quadrotor formation platform is introduced, and both simulation and experimental results are given. Section 6 concludes the whole work.

**Notations:** Throughout this paper, for simplicity of notation, let $0$ be zero matrices of appropriate size with zero vectors and zero number as special cases, and $\mathbf{1}_n$ be a column vector of size $N$ with $1$ as its elements. Denote by $I_n$ and $\mathbf{0}$ an identity matrix with dimension $n$ and the Kronecker product respectively. The superscripts $T$ and $H$ represent the transpose and the Hermitian adjoint of a matrix respectively.
2. Problem formulation and control law description

Consider a UAV swarm system consisting of $N$ UAVs. For each of these UAVs, since the trajectory dynamics has much larger time constants than the attitude dynamics, the formation control can be decoupled into an inner-loop control and an outer-loop control, where the inner-loop controller stabilizes the attitude and the outer-loop controller is used to drive the UAV towards the desired position (Bayezit & Fidan, 2013; Dong et al., 2015; Karimoddini et al., 2013). The schematic diagram of the two-loop structure for formation control is depicted in Fig. 1, where $h(t)$, $\phi(t)$, $\Phi(t)$, $\Gamma(t)$ and $\xi(t)$ represent the desired formation, desired attitude, control torque, position and velocity vectors, respectively. The current paper is mainly concerned with the formation control problems in the outer-loop, and the inner-loop can be controlled by the PD controller in Tayebi and McGilvray (2006). As shown in Wang et al. (2007), Wang and Xin (2013), Seo et al. (2012), and Dong et al. (2015), the outer-loop dynamics of UAV $i$, $i \in \{1, 2, \ldots, N\}$ can be approximately described by

$$
\begin{align*}
\dot{\xi}_i(t) &= v_i(t), \\
\dot{v}_i(t) &= u_i(t),
\end{align*}
$$

where $x_i(t) \in \mathbb{R}^n$, $v_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^n$ denote the position, velocity and control input vectors of UAV $i$ respectively. In the following, for the convenience of description, let $n=1$ if not otherwise specified. However, all the results hereafter can be directly extended to the higher dimensional case by using Kronecker product. The interaction topology among the $N$ UAVs can be described by an undirected graph $G = \{V, E, W\}$, where $V = \{v_1, v_2, \ldots, v_N\}$ represents the node set, $E \subseteq \{ (v_i, v_j) : v_i, v_j \in V \}$ is the edge set, and $W = [w_{ij}] \in \mathbb{R}^{N \times N}$ is a symmetric adjacency matrix with nonnegative elements $w_{ij}$. For all $i, j \in \{1, 2, \ldots, N\}$, UAVs $i$ and $j$ can be denoted by $v_i$ and $v_j$, and the interaction channel from UAV $i$ to UAV $j$ can be denoted by $e_{ij} = (v_i, v_j)$ in $E$. UAV $j$ is called a neighbor of UAV $i$ if there exists an edge $e_{ij}$. Let $N_i = \{v_i, v_{i1}, \ldots, v_{im}\}$ be the neighbor set of UAV $i$. The interaction strength $w_{ij}$ satisfies that $w_{ij} > 0$ if and only if $e_{ij} \in E$ and $w_{ij} = 0$. To describe the neighborhood relationship of the UAV swarm system, define the Laplacian matrix $L$ as $L = D - W$, where $D = \text{diag}(\sum_{j=1}^{N} w_{ij}, i = 1, 2, \ldots, N)$ is the in-degree matrix. An undirected graph $G$ is said to be connected if there is a path from each node to any other nodes. It should be pointed out that Laplacian matrix plays an important role in the cooperative control of swarm systems as it can reveal the topology requirement for swarm systems to achieve cooperative objectives. For instance, the second smallest eigenvalue of the Laplacian matrix called algebraic connectivity determines the speed of convergence for consensus or tracking algorithms (Kladis, 2014; Olfati-Saber et al., 2007).

The interaction topology of the UAV swarm system (1) can be switching. Let $S$ denote all the possible interaction topologies with an index set $\mathcal{S} \subseteq \mathbb{N}$, where $\mathbb{N}$ represents the set of natural numbers. Let $\sigma(t): [0, + \infty) \rightarrow \mathcal{S}$ be a switching signal which denotes the index of the topology at time $t$. Denote by $G_{\sigma(t)}$ and $L_{\sigma(t)}$ the interaction graph and the corresponding Laplacian matrix at $t$ respectively. Let $N_{\sigma(t)}$ be the neighbor set of agent $i$ at $\sigma(t)$. Throughout this paper, it is assumed that the admissible switching signal has a dwell time $T_0 > 0$, and all the interaction topologies in $S$ are connected.

![Fig. 1. Schematic diagram of the two-loop configuration for formation control.](image)

Define $\xi(t) = [x_i(t), v_i(t)]^T$, $B_1 = [1, 0]^T$ and $B_2 = [0, 1]^T$. Then UAV swarm system (1) can be rewritten as

$$
\dot{\xi}_i(t) = B_1 B_2^T \xi_i(t) + B_2 u_i(t).
$$

Denote by $h(t) = [h_1(t), h_2(t), \ldots, h_N(t)]^T \in \mathbb{R}^{N}$ the time-varying formation, where $h_i(t) = [h_{i1}(t), h_{i2}(t), \ldots, h_{in}(t)]^T$ ($i = 1, 2, \ldots, N$) are piecewise continuously differentiable vectors.

**Definition 1.** UAV swarm system (2) is said to achieve the time-varying formation $h(t)$ if there exists a vector-valued function $r(t) \in \mathbb{R}^n$ such that

$$
\lim_{t \to + \infty} \left( \xi_i(t) - h_i(t) - r(t) \right) = 0 \quad (i = 1, 2, \ldots, N),
$$

where $r(t)$ is called a formation reference function.

**Remark 1.** In protocol (3), the role of gain matrix $K_i$ is to specify the motion modes of the time-varying formation reference. If $K_i = 0$ and $h_{i0}(t) \equiv 0$, protocol (3) becomes the one that uses only neighboring relative information. Different from the protocols in Abdessameud and Tayebi (2011), Seo et al. (2012), Turpin et al. (2012), and Dong et al. (2015), the neighbors of each UAV in protocol (3) can be switching. It should be pointed out that collision avoidance and control input saturation are not considered in protocol (3). When designing a formation, it is necessary to keep a safety relative distance among UAVs and consider the constraint on the maneuver capability of each UAV.

Let $\xi(t) = [\xi_1(t), \xi_2(t), \ldots, \xi_N(t)]^T$, $h_1(t) = [h_{i1}(t), h_{i2}(t), \ldots, h_{in}(t)]^T$ and $h_2(t) = [h_{i1}(t), h_{i2}(t), \ldots, h_{in}(t)]^T$. Under protocol (3), UAV swarm system (2) can be expressed in a compact form as

$$
\xi(t) = (L_n \otimes (B_2 K_1 + B_2 K_2) - L_n \otimes (B_2 K_2)) \xi(t) - \left( L_n \otimes (B_2 K_1) - L_n \otimes (B_2 K_2) \right) h(t) + (L_n \otimes B_2) h_n(t).$$

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![Fig. 2. A triangle formation in XY plane with $N=3$.](image)
3. Problem transformation

In this section, a nonsingular transformation is constructed. By the transformation, the closed-loop UAV swarm system (4) is converted into two subsystems, one of which describes the formation error and the other one can be used to determine the formation reference. It is proven that the time-varying formation control problem of UAV swarm system (4) is equivalent to the asymptotic stability problem of the error subsystem. With the help of the transformation, the design of protocol (3) is transformed into choosing proper gain matrices to stabilize the error subsystem.

Let \( \xi_i(t) = \xi_i^a(t) - h_i(t) \) (i = 1, 2, ..., N) and \( \xi_i(t) = \begin{bmatrix} \xi_i^1(t) \\ \xi_i^2(t) \\ \vdots \\ \xi_i^N(t) \end{bmatrix} \). Then UAV swarm system (4) can be rewritten as follows:

\[
\dot{\xi}_i(t) = \left( I_N \otimes \left( B_i K_1 + B_i B_f^T \right) \right) \xi_i(t) \\
+ \left( I_N \otimes B_i \right) \left( h_i(t) - h_i(t) \right).
\]

Therefore, \( \lim_{t \to \infty} T(t) = h(t) - 1/\sqrt{N} I_N \otimes \left( \zeta^a(t) \right) = 0 \) if and only if \( \lim_{t \to \infty} \zeta(t) = 0 \). Because \( \dot{U} \otimes I_2 \) is nonsingular, from (9), UAV swarm system (4) achieves the time-varying formation \( h(t) \) if and only if \( \lim_{t \to \infty} \zeta(t) = 0 \) and \( \zeta(t) \) is a representation of the time-varying formation error. The conclusion of Lemma 2 can be obtained.

4. Main results

In this section, firstly, necessary and sufficient conditions for UAV swarm system (4) with switching interaction topologies to achieve the time-varying formation are presented. Then, an approach to determine the gain matrices in protocol (3) is given. Finally, an explicit expression of the formation reference function is proposed.

Denote by \( \lambda_{i,n}^o \) (i = 1, 2, ..., N) the eigenvalues of the Laplacian \( L_{i,n} \). Without loss of generality, it is assumed that \( \lambda_{i,n}^o \leq \lambda_{i,n}^1 \leq \cdots \leq \lambda_{i,n}^N \). Let \( \Lambda_{i,n}^o = \text{diag} \left( \lambda_{i,n}^2, \lambda_{i,n}^3, \cdots, \lambda_{i,n}^N \right) \). Furthermore, from Lemma 1, one sees that \( \lambda_{i,n}^0 \) is 0 with an associated eigenvector \( \tilde{u}_i = \sqrt{N} I \). Define \( \sigma_{\text{min}} = \min \left( \lambda_{i,n}^0 : \forall m \in \mathbb{N} ; i = 1, 2, \ldots, N \right) \).

Theorem 1. UAV swarm system (4) with switching interaction topologies achieves time-varying formation \( h(t) \) if and only if

(i) For all \( i \in \{ 1, 2, \ldots, N \} \)

\[
\lim_{t \to \infty} \left( h_{i,n}(t) - h_{i,m}(t) \right) = 0, \quad j \in N_{i,n}^i.
\]

(ii) The following switched linear system is asymptotically stable

\[
\theta(t) = \left( I_{N-1} \otimes \left( B_i K_1 + B_i B_f^T \right) \right) A_{i,n}(t) \otimes B_i \theta(t),
\]

where \( \theta(t) \) is the state of the system described by (14). .

Proof. Necessity: If UAV swarm system (4) with switching interaction topologies achieves the time-varying formation \( h(t) \), then from Lemma 2 and (7), one knows that

\[
\lim_{t \to \infty} \left( \dot{\zeta}_i(t) \right) = \left( \dot{\zeta}_i(t) \right) = 0,
\]

and the following system

\[
\dot{\xi}_i(t) = \left( I_N \otimes \left( B_i K_1 + B_i B_f^T \right) \right) \xi_i(t).
\]

Define \( \hat{h}_{i,n}(t) = \left( h_{i,n}(t), h_{i,n}(t), \ldots, h_{i,N-1,n}(t) \right)^T \) and \( \hat{h}_{i,m}(t) = \left( h_{i,m}(t), h_{i,m}(t), \ldots, h_{i,N-1,m}(t) \right)^T \). From (15) and (17), it follows

\[
\lim_{t \to \infty} \left( \hat{h}_{i,n}(t) - \hat{h}_{i,m}(t) - \theta(t) \right) = 0.
\]

Since \( \dot{U} \) is nonsingular, it is required to multiply both the sides of (18) by \( \dot{U}^{-1} \otimes I \) yields

\[
\lim_{t \to \infty} \left( \hat{h}_{i,n}(t) - \hat{h}_{i,m}(t) - \theta(t) \right) = 0.
\]

From (19), one sees that condition (i) is required.

Since \( \dot{U} \otimes L_{i,n} \) is symmetric, one can find an orthogonal matrix \( \tilde{U}_{i,n} \) satisfying \( \dot{U}_{i,n} \otimes L_{i,n} \tilde{U}_{i,n} = A_{i,n}(t) \). Let \( \theta(t) = \left( \dot{U}_{i,n}(t) \otimes I \right) \xi_i(t) \).

Therefore, \( \lim_{t \to \infty} T(t) - h(t) - 1/\sqrt{N} I_N \otimes \zeta^a(t) = 0 \) if and only if \( \lim_{t \to \infty} \zeta(t) = 0 \). Because \( \dot{U} \otimes I_2 \) is nonsingular, from (9), UAV swarm system (4) achieves the time-varying formation \( h(t) \) if and only if \( \lim_{t \to \infty} \zeta(t) = 0 \) and \( \zeta(t) \) is a representation of the time-varying formation error. The conclusion of Lemma 2 can be obtained.
Then system (16) can be transformed into
\[
\theta(t) = (L_{t-1} \otimes (B_t K_t + B_t B_t^T) - \Lambda_{t-1}) \otimes B_t K_t) \theta(t),
\]
that is, condition (ii) holds.

**Sufficiency:** If condition (i) holds, one knows that
\[
\lim_{t \to \infty} B_t \left( \left( h_{t-1}(t) - h_{t-1}(t) \right) \right)
= 0 \quad (t = 1, 2, \ldots, N; j \in \mathcal{N}_{e_{t-1}}),
\]
(20)

Therefore,
\[
\lim_{t \to \infty} \left( L_{t-1} \otimes B_t \left( h_{t}(t) - h_{t}(t) \right) \right) = 0.
\]
(21)

Substituting \( L_{t-1} = U \text{diag}(0, U^T \Lambda_{t-1}, U^T) \) into (21) and pre-multiplying both the sides of (21) by \( U^T \otimes B_t \), one has
\[
\lim_{t \to \infty} \left( U^T L_{t-1} U \otimes B_t \left( h_{t}(t) - h_{t}(t) \right) \right) = 0.
\]
(22)

Since \( U^T L_{t-1} U \) is nonsingular, it can be derived from (22) that
\[
\lim_{t \to \infty} \left( U \otimes B_t \left( h_{t}(t) - h_{t}(t) \right) \right) = 0.
\]
(23)

If condition (ii) holds, then system (16) is asymptotically stable. From (16), (23) and (7), one gets that \( \lim_{t \to \infty} \theta(t) = 0 \). From Lemma 2, one gets that Theorem 1 holds.

**Remark 2.** From condition (i) in Theorem 1, it is evident that not all the time-varying formations can be achieved by UAV swarm system (4). The feasible formation must satisfy equation (13). This requirement is reasonable. For example, it is impossible for three UAVs to achieve a formation that the position components keep a fixed triangle shape while the values of the velocity components are not equal to each other. Moreover, from (7), (14) and (16), one can obtain the relationship among the states \( \theta(t) \), \( \xi(t) \) and the formation error \( \zeta(t) \); that is, \( \theta(t) = (\Omega_{t-1} \otimes I) \zeta(t) \) and that \( \zeta(t) \) is equivalent to \( \xi(t) \) in the case where Eq. (15) holds.

**Theorem 2.** If condition (i) in Theorem 1 holds, then UAV swarm system (2) achieves time-varying formation \( h(t) \) by protocol (3) with \( K_t = (2 \lambda_{\min})^{-1} B_t^T P \), where \( P \) is the positive definite solution to the following algebraic Riccati equation:
\[
P \left( B_t K_t + B_t B_t^T \right) + \left( B_t K_t + B_t B_t^T \right)^T P - P B_t B_t^T P + I = 0.
\]
(24)

**Proof.** Consider the stability of system (14). Construct the following Lyapunov candidate function:
\[
V(t) = \theta^T(t) \left( \Omega_{t-1} \otimes I_2 \right)^T (I_{n-1} \otimes P) \left( \Omega_{t-1} \otimes I_2 \right) \theta(t).
\]
(25)

Because \( \dot{\xi}(t) \) is continuously differentiable, if condition (i) in Theorem 1 holds, from (9) one knows
\[
\dot{\xi}(t) = \left[ \left( \Omega_{t-1} \otimes I_2 \right) \theta(t) \right].
\]
(26)

Therefore, \( \dot{\xi}(t) \) is continuously differentiable, so is \( V(t) \). Since \( \left( \Omega_{t-1} \otimes I_2 \right)^T \dot{\xi}(t) = \left( \Omega_{t-1} \otimes I_2 \right) V(t) \), one can rewrite as
\[
V(t) = \theta^T(t) (I_{n-1} \otimes P) \theta(t).
\]
(27)

Taking the derivative of \( V(t) \) with respect to \( t \) along the trajectory of system (14), one has
\[
\dot{V}(t) = \theta^T(t) \left( I_{n-1} \otimes \Xi - \Lambda_{t-1} \right) \left( K_t B_t^T P + P B_t K_t \right) \theta(t),
\]
where \( \Xi = (B_t K_t + B_t B_t^T)^T P + P (B_t K_t + B_t B_t^T) \); that is,
\[
\dot{V}(t) = \sum_{i=2}^{N} \theta_i^T(t) \left( \Xi - \lambda_{\min} \left( K_t B_t^T P + P B_t K_t \right) \right) \theta_i(t).
\]
(28)

Substituting \( K_t = (2 \lambda_{\min})^{-1} B_t^T P \) and \( \Xi = PB_t B_t^T P - I \) into (29), one has
\[
\dot{V}(t) = \sum_{i=2}^{N} \theta_i^T(t) \left( -I + \left( 1 - \lambda_{\min} \left( K_t B_t^T P + P B_t K_t \right) \right) \theta_i(t) \right).
\]
(29)

From the definition of \( \lambda_{\min} \), one has \( \lambda_{\min} \leq \lambda_{\min} \), for any \( \sigma(t) \in I \) and \( i \in \left( 2, 3, \ldots, N \right) \), which means that \( 1 - \lambda_{\min} \leq 0 \). Since \( 1 - \lambda_{\min} \leq 0 \), one gets that \( V(t) \leq 0 \). Note that \( \dot{T}_d > 0, V(t) \equiv 0 \) if and only if \( \theta_i(t) \equiv 0 \) \( i \in \left( 2, 3, \ldots, N \right) \), which means that \( \theta(t) \equiv 0 \). Therefore, condition (ii) in Theorem 1 holds. From Theorem 1, one knows that UAV swarm system (2) achieves time-varying formation \( h(t) \) by protocol (3). This completes the proof.

If UAV swarm system (4) achieves time-varying formation, then an explicit expression of the formation reference function can be obtained to describe the macroscopic movement of the whole UAV formation as follows.

**Corollary 1.** If UAV swarm system (4) achieves time-varying formation under switching topologies, then the formation reference function \( r(t) \) has the following form:
\[
\lim_{t \to \infty} \left( r(t) - r_0(t) - r_h(t) \right) = 0,
\]
where
\[
r_0(t) = \left( B_2 K_2 + B_2 B_2^T \right)^T P \sum_{i=1}^{N} \frac{1}{N} \left( \Omega_{i-1} \otimes I_2 \right) \xi(0),
\]
\[
r_h(t) = \frac{1}{N} \sum_{i=1}^{N} h_i(t)
+ \int_0^t e^{B_2 K_2 + B_2 B_2^T (t-s)} B_2 K_2 \sum_{i=1}^{N} \frac{1}{N} \left( \Omega_{i-1} \otimes I_2 \right) \xi(s) - \frac{1}{N} \sum_{i=1}^{N} h_i(s) \right) ds.
\]

**Proof.** If UAV swarm system (4) achieves time-varying formation under switching topologies, then \( \lim_{t \to \infty} \theta(t) = 0 \). From (12), one has
\[
\lim_{t \to \infty} \left( \xi(t) - \frac{1}{\sqrt{N}} \mathbf{1}_N \otimes \zeta(t) \right) = 0.
\]
(30)

It can be shown that
\[
\zeta(0) = \frac{1}{\sqrt{N}} \left( \mathbf{1}_N \otimes I_2 \right) \xi(t) = \frac{1}{\sqrt{N}} \left( \mathbf{1}_N \otimes I_2 \right) \left( \xi(0) - h(0) \right),
\]
and
\[
\int_0^t e^{B_2 K_2 + B_2 B_2^T (t-s)} \frac{1}{\sqrt{N}} \left( \mathbf{1}_N \otimes I_2 \right) \left( h_i(s) - h_i(s) \right) ds
= - \frac{1}{N} \sum_{i=1}^{N} h_i(t) + \frac{1}{\sqrt{N}} e^{B_2 K_2 + B_2 B_2^T t} \frac{1}{N} \sum_{i=1}^{N} h_i(0)
+ \int_0^t e^{B_2 K_2 + B_2 B_2^T (t-s)} B_2 \sum_{i=1}^{N} \frac{1}{N} \left( \Omega_{i-1} \otimes I_2 \right) \xi(s) - \frac{1}{N} \sum_{i=1}^{N} h_i(s) \right) ds.
\]
(32)

From (6) and (30)–(32), the conclusion of Corollary 1 can be obtained.

**Remark 3.** From Corollary 1, one sees that switching topologies have no effect on the formation reference function. Moreover, one can always find the appropriate \( K_t \) to specify the motion modes of the formation reference by placing the eigenvalues of \( B_2 K_2 + B_2 B_2^T \) at desired locations in the complex plane.

Based on the above results, a procedure for determining \( K_t \) and \( K_t \) can be summarized as follows:

**Step 1:** For a given formation, check the formation feasibility condition (13), if it is satisfied then continue; else this formation
cannot be realized by UAV swarm system (2) under protocol (3) and stop.

Step 2: Choose $K_1$ to specify the motion modes of the formation reference by assigning the eigenvalues of $B_t K_1 + B_t B_t^T$ at desired locations in the complex plane.

Step 3: Solve the algebraic Riccati equation (24) for $P$ and then choose $K_0 = (2\lambda_{\min})^{-1} B_t^T P$.

5. Simulation and experimental results

In this section, firstly, a quadrotor formation platform is introduced. Then both a numerical simulation and a practical experiment are carried out on the quadrotor formation platform with four quadrotors in Example 1. Moreover, to better demonstrate the scalability of the obtained results, a large scale example with 10 quadrotors is given as Example 2. Due to the quantity limitation of the quadrotor UAVs in our lab, experimental results are not presented in Example 2.

5.1. Quadrotor formation platform

Fig. 3 shows the quadrotor formation platform which comprises one ground control station (GCS) and four quadrotors with flight control system (FCS). The tip-to-tip wingspan of the quadrotor is 65 cm, and the weight is 1600 g. The maximum take-off weight of each quadrotor is 1800 g, and the maximum flight time is about 12 min.

The FCS is developed based on a TMS320F2833S DSP running at 135 MHz. Three one-axis gyroscopes, a three-axis magnetometer and a three-axis accelerometer are employed by the FCS to estimate the attitude and acceleration of the quadrotor. The position and the velocity of each quadrotor are measured by the global positioning system (GPS) module with an accuracy of 1.2 m circular error probable (CEP) at a rate of 10 Hz. When the quadrotor is near the ground, the height is measured by an ultrasonic range finder. A 2G micro SD card is used to record the main flight parameters onboard. The wireless communications among quadrotors and the GCS are implemented by Zigbee modules. Control commands are sent to a specified quadrotor or broadcasted to all quadrotors through the Zigbee network. The states of all quadrotors are sent to the GCS and monitored by the real-time display module on the GCS. During the formation, each quadrotor need neither the control of the remote controller nor the control of the GCS. However, to deal with the emergency situation, an RC receiver is kept on each quadrotor. Fig. 4 illustrates the hardware structure of the quadrotor system.

5.2. Simulations and experiments

The formation control of the quadrotor swarm system is implemented in the horizontal plane ($n=2$); that is, the movements of the quadrotors along X- and Y-axes are controlled by the formation protocol (3) with a rate of 5 Hz. The height of each quadrotor is specified to be constant. The pitch, roll and yaw angles of each quadrotor are controlled by three decoupled PD controllers shown in Tayebi and McGilvray (2006) with a rate of 500 Hz respectively as the inner loop. Note that the movements of each quadrotor along X- and Y-axes are decoupled. Using the Kronecker product, the dynamics of the quadrotor swarm system in the horizontal plane can be described by (2) with $\tilde{x}_i(t) = [x_{iX}(t), y_{iX}(t), x_{iY}(t), y_{iY}(t)]^T$, $u_{iX}(t) = [u_{iX}(t), u_{iY}(t)]^T$, $\tilde{h}_i(t) = [h_{iX}(t), h_{iYX}(t), h_{iY}(t), h_{iYY}(t)]^T$ ($i = 1, 2, ..., N$), $h_{i} - h_{i} \in [0, 0.1]^2$, $B_1 = I_2 \otimes [0, 1]^2$, where $j = 1, 2, ..., N$, $x_{iX}(t)$, $y_{iX}(t)$, $u_{iX}(t)$, $h_{iX}(t)$, $h_{iYX}(t)$ and $h_{iY}(t)$ are the position, velocity, control input, desired formation components of quadrotor $i$ along X- and Y-axes, respectively.

From the formation protocol (3), one sees that only the position and the velocity of each quadrotor and its neighbors are required to construct the controller. In the experiment, the position and the velocity of each quadrotor are obtained by the complementary filter (refer to Brown & Hwang, 1996; Euston, Coote, Mahony, Jonghyuk, & Hamel, 2008 for more details) which combines the accelerometer measurement with the GPS measurement. The neighboring position and velocity are transmitted by the Zigbee network. Due to that the movements of each quadrotor along X- and Y-axes are decoupled, the controllers of each quadrotor along X- and Y-axes can be designed separately. For simplicity, it is assumed that all interaction topologies in this section are 0–1 weighted.

Example 1. Simulation and experiment with four quadrotors

Consider the following time-varying formation:

$$h_i(t) = \begin{bmatrix}
    r \cos(\omega t + (i - 1)\pi/2) \\
    - \omega t \sin(\omega t + (i - 1)\pi/2) \\
    r \sin(\omega t + (i - 1)\pi/2) \\
    \omega t \cos(\omega t + (i - 1)\pi/2)
\end{bmatrix},$$

$i = 1, 2, 3, 4$,

where $r = 10$ m and $\omega = 0.1$ rad/s. For simplicity, assume that there exist four interaction topologies in set $\mathcal{S}$ (as shown in Fig. 5). The interaction topology is randomly chosen from $\mathcal{S}$ with interval $T_d = 10$ s. If $h(t)$ is achieved by the quadrotor swarm system under...
switching interaction topologies, then both the positions and velocities of the four quadrotors locate at the vertexes of a rotating regular square respectively in the \( XY \) plane. It can be verified that condition (i) in Theorem 1 is satisfied. Due to the limitation of flight space and the requirement of performing the experiment within a visual range, the motion modes of the formation reference \( r(t) \) are designed to be stable by choosing \( K_I = 0.812 \) to assign the eigenvalues of \( B_K B_K^T \) at \( j0.4, 0.9165 \), \( j0.4, 0.9165 \) and \( j0.4, 0.9165 \) respectively, which means that \( \lambda_{\min} = 0.5858 \). Using the approach in Theorem 2, one can obtain the matrix \( P \) as

\[
P = I_2 \otimes \begin{bmatrix} 1.4219 & 0.4142 \\ 0.4142 & 0.7711 \end{bmatrix}
\]

and matrix \( K_S = I_2 \otimes [0.3535, 0.6582] \) to ensure that the UAV swarm system can achieve the desired formation.

Choose the initial states of four quadrotors as \( \xi(0) = [9.84, -0.11, 0.19, 0.07]^T \), \( \xi(0) = [-0.41, 0.04, 10.51, 0.22]^T \), \( \xi(0) = [-10.47, 0.08, 0.48, 0.02]^T \) and \( \xi(0) = [-0.93, -0.08, -9.11, -0.25]^T \). Figs. 6 and 7 show the state trajectories of the four quadrotors and the formation reference in the simulation and experiment within
$t = 126$ s respectively, where the initial states of the four quadrotors and the formation reference are marked by circles and the final states are denoted by asterisks, diamonds, triangles, squares and pentagrams respectively. Define the energy of the formation error as $\varepsilon(t)$. Fig. 8 depicts the energy curve of the formation error $\varepsilon(t)$ in both the simulation and the experiment. Figs. 9 and 10
show the control inputs of the four quadrotors along $X$- and $Y$-axes in both the simulation and the experiment, respectively. Fig. 11 shows a captured image of four quadrotors in the formation flight. From Figs. 7 to 11, one sees that the quadrotor swarm system achieves the predefined time-varying formation under switching interaction topologies in both simulation and experiment. The video of the experiment can be found at http://v.youku.com/v_show/id_XNjY3OTE4OTI0.html or https://www.youtube.com/watch?v=9cVvdA7Dv3M. It should be pointed out that due to the existence of external disturbances, sensor errors and communication delays in the experiment, there are certain small errors in Fig. 11.

**Fig. 11.** Formation flight image in the experiment.

**Fig. 12.** Switching interaction topologies in Example 2. (a) $G_5$, (b) $G_6$, (c) $G_7$ and (d) $G_8$.

**Fig. 13.** State trajectories of the 10 quadrotors and $r(t)$ in Example 2. (a) Positions and (b) velocities.

**Fig. 14.** Energy curve of the formation error $\zeta(t)$ in Example 2.
the experimental results in comparison with the simulation.

For example, the formation reference $r(t)$ in Fig. 6 is stationary while in Fig. 7 it moves in a small range, and the energy of the formation error $\xi(t)$ converges to zero in Fig. 8(a) while in Fig. 8(b) it converges to a small error bound. These errors occurring in the experiment are inevitable and reasonable.

**Example 2.** Simulation with ten quadrotors

Consider a quadrotor swarm system with ten quadrotors. The desired time-varying formation for the 10 quadrotors is specified by

$$ h_i(t) = \begin{bmatrix} r \sin (\omega t + (i - 1)\pi/5) \\ r \cos (\omega t + (i - 1)\pi/5) \\ r \cos (\omega t + (i - 1)\pi/5) \\ -a\omega \sin (\omega t + (i - 1)\pi/5) \end{bmatrix} \quad (i = 1, 2, \ldots, 10), $$

where $r=20$ m and $\omega=0.15$ rad/s. Assume that there exist four interaction topologies in set $S$ which are shown in Fig. 12. The interaction topologies are randomly chosen from $S$ with interval $T_d = 6$ s. If $h(t)$ is achieved by the quadrotor swarm system under switching interaction topologies, then both the positions and velocities of the 10 quadrotors will form a regular decagon while keep rotating around the formation reference respectively in the horizontal plane. It can be verified that condition (i) in Theorem 1 is satisfied. Different from Example 1, the motion modes of the formation reference $r(t)$ are designed to be oscillating by choosing $K_i = l_2 \otimes [ -0.36, 0]$ to assign the eigenvalues of $B_iK_i + B_iB_i^T$ at $-0.6j, -0.6j, 0.6j$ and $0.6j$. In this case, the formation reference will move periodically. From Fig. 12, one gets the smallest nonzero eigenvalues of the four Laplacian matrices as $0.3820, 0.2087, 0.1487$ and $0.0979$ respectively, which means that $\lambda_{\min} = 0.0979$. Using the approach in Theorem 2, one can obtain the matrix $P$ as

$$ P = l_2 \otimes \begin{bmatrix} 1.6485 & 0.7028 \\ 0.7028 & 1.5510 \end{bmatrix} $$

and the matrix $K_0 = l_2 \otimes [3.5895, 7.9214]$.

Choose the initial states of the 10 quadrotors as

$$ \xi_i(0) = [1, 2.5, 18.3, -0.6]^T, \quad \xi_2(0) = [12.1, 1.9, 16.7, -2.2]^T, \quad \xi_3(0) = [20.3, 1.4, 6.6, -3.5]^T, \quad \xi_4(0) = [21.7, -1.3, -6.3, -2.1]^T, $$

$$ \xi_5(0) = [12.7, -3.4, -14.1, -2.7]^T, \quad \xi_6(0) = [1.4, -3.5, 20.8, -0.8]^T, \quad \xi_7(0) = [-8.7, -3.4, -17.4, 2.6]^T, \quad \xi_8(0) = [-14.6, -0.5, -5.2, 2]^T, $$

$$ \xi_9(0) = [-13.9, 1.2, 8.3, 1.8]^T, \quad \xi_{10}(0) = [-12.6, 1.5, 15.6, 0.3]^T. $$

**Fig. 13** shows the state trajectories of the ten quadrotors and the formation reference in the simulation within $t=60$ s, where the initial states of the quadrotors and the formation reference are marked by circles and the final states are denoted by points and pentagrams, respectively. **Fig. 14** depicts the energy curve of the formation error $\xi(t)$ in the simulation. **Fig. 15** shows the control inputs of the 10 quadrotors along X- and Y-axes, respectively. From **Fig. 13**, the following phenomena can be found: (1) both the position and velocity components of the 10 quadrotors form the regular decagon formation; (2) the regular decagon formation keeps rotating around the formation reference $r(t)$; and (3) the states of formation reference move periodically, which means that the whole time-varying formation moves periodically. Therefore, the desired time-varying formation is achieved by the 10 quadrotors under the switching topologies. Moreover, due to the fact that each quadrotor only uses the neighboring information and the calculation complexity for determining the gain matrices in the protocol (3) is independent with the number of quadrotors, the obtained results have good scalability.

**6. Conclusions**

Time-varying formation control problems for UAV swarm systems with switching interaction topologies were investigated. A distributed time-varying formation protocol was proposed. Necessary and sufficient conditions for UAV swarm systems with switching interaction topologies to achieve desired time-varying formations were presented. An approach to determine the gain matrices in the formation protocol was proposed based on common Lyapunov functional approach and algebraic Riccati equation technique. An explicit expression of the formation reference function and approaches to assign the motion modes of the formation reference were given. The theoretical results were demonstrated by time-varying formation flight experiments with four quadrotors in outdoor environment. Based on this result, it is
of interest to further study formation control problems for UAV swarm systems with nonlinear dynamics. For practical implementation of the protocol, it is also necessary to consider collision avoidance and control input saturation in the protocol.

References


